On Clique Convergences of Graphs

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Abstract

Let G be a graph and $\mathcal{K}_G$ be the set of all cliques of G, then the clique graph of G denoted by $K(G)$ is the graph with vertex set $\mathcal{K}_G$ and two elements $Q_i, Q_j \in \mathcal{K}_G$ form an edge if and only if $Q_i \cap Q_j \neq \emptyset$. Iterated clique graphs are defined by $K^0(G) = G$, and $K^n(G) = K(K^{n-1}(G))$ for $n > 0$. In this paper we determine the number of cliques in $K(G)$ when $G = G_1 + G_2$, prove a necessary and sufficient condition for a clique graph $K(G)$ to be complete when $G = G_1 + G_2$, give a characterization for clique convergence of the join of graphs and if G_1, G_2 are Clique-Helly graphs different from K_1 and $G = G_1 \Box G_2$, then $K^2(G) = G$.

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1 Introduction

Given a simple graph $G = (V, E)$, not necessarily finite, a clique in G is a maximal complete subgraph in G. Let G be a graph and $\mathcal{K}_G$ be the set of all cliques of G, then the clique graph operator is denoted by K and the clique graph of G is denoted by $K(G)$. Where $K(G)$ is the graph with vertex set $\mathcal{K}_G$ and two elements $Q_i, Q_j \in \mathcal{K}_G$ form an edge if and only if $Q_i \cap Q_j \neq \emptyset$. Clique graph was introduced by Hamelink in 1968 [1]. Iterated clique graphs are defined by $K^0(G) = G$, and $K^n(G) = K(K^{n-1}(G))$ for $n > 0$(see [2, 5, 6]).

Definition 1.1 *A graph G is said to be K-periodic if there exists a positive integer n such that $G \cong K^n(G)$ and the least such integer is called the K-periodicity of G, denoted K-per (G).*

Definition 1.2 *A graph G is said to be K-Convergent if $\{K^n(G) : n \in \mathbb{N}\}$ is finite, otherwise G is K-Divergent (see [4]).*

Definition 1.3 *A graph H is said to be K-root of a graph G if $K(H) = G$.*

If G is a clique graph then one can observe that, the set of all K- roots of G is either empty or infinite.

Definition 1.4 *[5] A graph G is a Clique-Helly Graph if the set of cliques has the Helly-Property. That is, for every family of pairwise intersecting cliques of the graph, the total intersection of all these cliques should be non-empty also.*

Definition 1.5 *Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. Then their join $G_1 + G_2$ is obtained from the disjoint union by adding all possible edges between vertices of G_1 and G_2.*

Definition 1.6 *The Cartesian product of two graphs G and H, denoted $G\Box H$, is a graph with vertex set $V(G\Box H) = V(G) \times V(H)$,i.e., the set $\{(g, h)|g \in G, h \in H\}$. The edge set of $G\Box H$ consists of all pairs $[(g_1, h_1), (g_2, h_2)]$ of vertices with $[g_1, g_2] \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $[h_1, h_2] \in E(H)$ (see [3] page no 3).*

In this paper we determine the number of cliques in $K(G)$ when $G = G_1 + G_2$, prove a necessary and sufficient condition for a clique graph $K(G)$ to be complete when $G = G_1 + G_2$,give a characterization for clique convergence of the join of graphs and if G_1, G_2 are Clique-Helly graphs different from K_1 and $G = G_1\Box G_2$, then $K^2(G) = G$.

2 Results

ne can observe that the clique graph of a complete graph and star graph are always complete. Let G be a graph with n vertices and having a vertex of degree $n-1$, then the clique graph of G is also complete.

Theorem 2.1 *Let G_1, G_2 be two graphs and $G = G_1 + G_2$, then X is a clique in G_1 and Y is clique in G_2 if and only if $X + Y$ is a clique in $G_1 + G_2$.*

Proof: Let $G = G_1 + G_2$ and X be a clique in G_1 and Y be a clique in G_2. Suppose that $X + Y$ is not a maximal complete subgraph in $G_1 + G_2$, then there is a maximal complete subgraph (clique) Q in $G_1 + G_2$ such that $X + Y$ is a proper subgraph of Q. Since $X + Y$ is a proper subgraph of Q, there is a vertex v in Q which is not in $X + Y$ and v is adjacent to every vertex of $X + Y$, then by the definition of $G_1 + G_2$, v should be in either G_1 or G_2. Suppose v is in G_1, then the induced subgraph of $V(X) + \{v\}$ is complete in G_1, which is a contradiction as X is maximal. Therefore $X + Y$ is the maximal complete subgraph (clique) in $G_1 + G_2$.

Conversely, let Q is a clique in $G_1 + G_2$. Suppose that $Q \neq X + Y$ where X is a clique in G_1 and Y is a clique in G_2. If $Q \cap G_1 = \emptyset$, then Q is a subgraph of G_2. This implies that Q is a clique in G_2 as Q is a clique in G. Let v be a vertex of G_1. Then by the definition of $G_1 + G_2$, one can observe that the induced subgraph of $V(Q) \cup \{v\}$ is complete in G, which is a contradiction as Q is a maximal complete subgraph. Therefore $Q \cap G_1 \neq \emptyset$. Similarly we can prove that $Q \cap G_2 \neq \emptyset$. Let X be the induced subgraph of G with vertex set $V(Q) \cap V(G_1)$ and Y be the induced subgraph of G with vertex set $V(Q) \cap V(G_2)$, then $Q = X + Y$. Since Q is a maximal complete subgraph of G, X and Y should be maximal complete subgraphs in G_1 and G_2 respectively. Otherwise, if X is not a maximal complete subgraph in G_1 then there is a maximal complete subgraph X' in G_1 such that X is subgraph of X', and this implies that $X + Y$ is a subgraph of $X' + Y$ and $X' + Y$ is complete, which is a contradiction. Therefore X and Y are maximal complete subgraphs (cliques) in G_1 and G_2 respectively. ■

Corollary 2.2 *Let G_1, G_2 be two graphs and $G = G_1 + G_2$. If n, m are the number of cliques in G_1, G_2 respectively, then G has nm cliques.*

Proof: Let $G = G_1 + G_2$, $\mathcal{K}_{G_1} = \{X_1, X_2, \ldots, X_n\}$ be the set of all cliques of G_1 and $\mathcal{K}_{G_2} = \{Y_1, Y_2, \ldots, Y_m\}$ be the set of all cliques of G_2. Then by Theorem 2.1 it follows that $\mathcal{K}_G = \{X_i + Y_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is the set of all cliques of G. Since G_1 has n, G_2 has m number of cliques, $G_1 + G_2$ has nm number of cliques. ■

In the following result we give a necessary and sufficient condition for a clique graph $K(G)$ to be complete when $G = G_1 + G_2$.

Theorem 2.3 *Let G_1, G_2 be two graphs. If $G = G_1 + G_2$, then $K(G)$ is complete if and only if either $K(G_1)$ is complete or $K(G_2)$ is complete.*

Proof: Let $G = G_1 + G_2$ and $K(G)$ be complete. Suppose that neither $K(G_1)$ nor $K(G_2)$ is complete, then there exist two cliques X, X' in G_1 and two cliques Y, Y' in G_2 such that $X \cap X' = \emptyset$ and $Y \cap Y' = \emptyset$. By Theorem 2.1 it follows that $X + Y, X' + Y'$ are cliques in G. Since $X \cap X'$ and $Y \cap Y'$ are empty, it follows that $\{X + Y\} \cap \{X' + Y'\} = \emptyset$, which is a contradiction as $K(G)$ is complete.

Conversely, suppose that $K(G_1)$ is complete and $\mathcal{K}_{G_1} = \{X_1, X_2, \ldots, X_n\}$, $\mathcal{K}_{G_2} = \{Y_1, Y_2, \ldots, Y_m\}$. By Corollary 2.2, it follows that G has exactly nm number of cliques. Let $\mathcal{K}_G = \{Q_{ij} : Q_{ij} = X_i + Y_j \text{ for } i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\}$ be the set of all cliques of G. Then Q is the vertex set of $K(G)$. Arranging the elements of $\mathcal{K}_G$ in the matrix form $M = [m_{ij}]$ where $m_{ij} = Q_{ij}$, we have

$$M = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & \cdots & Q_{1m} \\ Q_{21} & Q_{22} & Q_{23} & \cdots & Q_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & Q_{n3} & \cdots & Q_{nm} \end{pmatrix}.$$

Let Q_{ij}, Q_{kl} be any two elements in M. Since $Q_{ij} = X_i + Y_j$, $Q_{kl} = X_k + Y_l$, it follows that X_i, X_k are cliques in G_1. Since $K(G_1)$ is complete, $X_i \cap X_k \neq \emptyset$ and then $Q_{ij} \cap Q_{kl} \neq \emptyset$. Therefore Q_{ij}, Q_{kl} are adjacent in $K(G)$. Hence $K(G)$ is complete. ■

Lemma 2.4 *Let G_1, G_2 be two graphs and $G = G_1 + G_2$. If $K(G_1)$, $K(G_2)$ are not complete, then for every clique in $K(G_1)$ there is a clique in $K(G)$.*

Proof: Let $G = G_1 + G_2$ be a graph such that $K(G_1)$ and $K(G_2)$ are not complete. Let $V(K(G_1)) = \{X_i : X_i \text{ is a clique in } G_1, 1 \leq i \leq n\}$ and $V(K(G_2)) = \{Y_j : Y_j\text{t is a clique in } G_2, 1 \leq j \leq m\}$, then by Theorem 2.1 it follows that $V(K(G)) = \{X_i + Y_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. Let Q be a clique of size l in $K(G_1)$ and $V(Q) = \{X_{Q_1}, X_{Q_2}, \ldots, X_{Q_l}\}$ where X_{Q_i} is a clique in G_1 for $1 \leq i \leq l$. Let $A_Q = \{X_{Q_i} + Y_j : 1 \leq i \leq l, 1 \leq j \leq m\}$. Then clearly A_Q is subset of $V(K(G))$.

Let $X_{Q_1} + Y_1$, $X_{Q_2} + Y_2$ be two elements in A_Q. Since X_{Q_1}, X_{Q_2} are the vertices of the clique Q of $K(G_1)$, we have $X_{Q_1} \cap X_{Q_2} \neq \emptyset$. Therefore $\{X_{Q_1} + Y_1\} \cap \{X_{Q_2} + Y_2\} \neq \emptyset$. Hence the intersection of any two elements in A_Q is nonempty. Then, it follows that the elements of A_Q form a complete subgraph in $K(G)$. Suppose that it is not a maximal complete subgraph in $K(G)$. Then there is a vertex, say $X_1 + Y_1$ in $K(G)$ which is not in A_Q and $X_1 + Y_1$ is adjacent with every vertex of A_Q. Since $K(G_2)$ is not complete there exists a vertex say Y_2 in $K(G_2)$ such that Y_2 is not adjacent to Y_1 in $K(G_2)$. Since Q is a clique in $K(G_1)$ and $K(G_1)$ is not complete, there is a vertex say X_{Q_1} in $V(Q)$ which is not adjacent to X_1 in $K(G_1)$. By the definition of A_Q one

can see that $X_{Q_1} + Y_2$ is an element of A_Q. Therefore $\{X_{Q_1} + Y_2\} \cap \{X_1 + Y_1\} = \emptyset$, which is a contradiction. Thus A_Q is a maximal complete subgraph in $K(G)$. Hence for every clique in $K(G_1)$ there is a clique in $K(G)$. ■

Similarly for every clique in $K(G_2)$, there is a clique in $K(G)$.

Lemma 2.5 *Let G_1, G_2 be two graphs and $G = G_1 + G_2$. If $K(G_1)$, $K(G_2)$ are not complete, then for every clique in $K(G)$ there is a clique, either in $K(G_1)$ or in $K(G_2)$ but not in both.*

Proof: Let $G = G_1 + G_2$ be a graph such that $K(G_1)$ and $K(G_2)$ are not complete. Let $V(K(G_1)) = \{X_i : X_i \text{ is a clique in } G_1, 1 \leq i \leq n\}$ and $V(K(G_2)) = \{Y_j : Y_j \text{ is a clique in } G_2, 1 \leq j \leq m\}$, then by Theorem 2.1, $V(K(G)) = \{X_i + Y_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. One can observe that for every X_i in $V(K(G_1))$, the vertices $X_i + Y_1, X_i + Y_2, \ldots, X_i + Y_m$ form a complete graph in $K(G)$, $1 \leq i \leq n$. Similarly, for every Y_j in $V(K(G_2))$, the vertices $X_1 + Y_j, X_2 + Y_j, \ldots, X_n + Y_j$ form a complete graph in $K(G)$, $1 \leq j \leq m$. Therefore every clique in $K(G)$ is of size ln or lm. Let Q be a clique of size lm in $K(G)$ and $V(Q) = \{X_{Q_1} + Y_1, X_{Q_1} + Y_2, \ldots, X_{Q_1} + Y_m, X_{Q_2} + Y_1, X_{Q_2} + Y_2, \ldots, X_{Q_2} + Y_m, \ldots, X_{Q_l} + Y_1, X_{Q_l} + Y_2, \ldots X_{Q_l} + Y_m\}$ where X_{Q_i}, for $1 \leq i \leq l$ is the clique in G_1. Define $A_Q = \{X_{Q_i} : 1 \leq i \leq l\}$. Clearly A_Q is a subset of $V(K(G_1))$.

Let X_{Q_1} and X_{Q_2} be two elements of A_Q. Since $K(G_2)$ is not complete, there exists a vertex, say Y_2 in $K(G_2)$ such that Y_2 is not adjacent to Y_1 in $K(G_2)$, this implies that $Y_1 \cap Y_2 = \emptyset$. Since $X_{Q_1} + Y_1, X_{Q_2} + Y_2$ are the vertices of the clique Q of G, $\{X_{Q_1} + Y_1\} \cap \{X_{Q_2} + Y_2\} \neq \emptyset$. Therefore $X_{Q_1} \cap X_{Q_2} \neq \emptyset$. Hence the intersection of any two elements in A_Q is nonempty. It follows that the elements of A_Q form a complete subgraph in $K(G_1)$. Suppose that it is not a maximal complete subgraph in $K(G_1)$, then there is a vertex say X_1 in $K(G)$ which is not in A_Q and X_1 is adjacent with every vertex in A_Q, this implies that $X_1 \cap X_{Q_i} \neq \emptyset$, $1 \leq i \leq l$. Since X_1 is not in A_Q, the vertex $X_1 + Y_1$ is not in Q. Since X_1 is adjacent with every element in A_Q, $\{X_1 + Y_1\} \cap \{X_{Q_i} + Y_j\} \neq \emptyset$ for every i, j, $1 \leq i \leq l$, $1 \leq j \leq m$. This implies that the vertex $X_1 + Y_1$ is adjacent to every vertex of Q in $K(G)$, which is a contradiction as Q is maximal in $K(G)$. Therefore the elements of A_Q form a maximal complete subgraph (clique) in $K(G_1)$. Hence for every clique of size lm in $K(G)$ there is a clique of size l in $K(G_1)$. Similarly we can prove that if the clique in $K(G)$ is of size ln, then there is a clique of size l in $K(G_2)$. ■

Lemma 2.6 *Let G_1, G_2 be two graphs and $G = G_1 + G_2$. If $K(G_1)$, $K(G_2)$ are not complete, then the number of cliques in $K(G)$ is the sum of the number of cliques in $K(G_1)$ and $K(G_2)$.*

Proof: Proof of this Lemma follows by the Lemmas 2.4 and 2.5. ∎

By the definition of a clique graph, cliques of $K(G)$ are the vertices of $K^2(G)$. By Lemmas 2.4, 2.5 and 2.6 it follows that there is a one to one correspondence between $V(K^2(G))$ and $V(K^2(G_1)) \cup V(K^2(G_2))$ where $G = G_1 + G_2$. By the definition of a clique graph, cliques of G are the vertices of $K(G)$. By Corollary 2.2 it follows that, if $|V(K(G_1))| = n$ and $|V(K(G_2))| = m$ then $|V(K(G))| = nm$. Therefore $K(G) \neq K(G_1) + K(G_2)$.

Theorem 2.7 *Let G_1, G_2 be two graphs and $G = G_1 + G_2$. If $K(G_1)$, $K(G_2)$ are not complete, then $K^2(G) = K^2(G_1) + K^2(G_2)$.*

Proof: Let $G = G_1 + G_2$ be a graph such that $K(G_1)$ and $K(G_2)$ are not complete. Let $X_1, X_2, \ldots, X_n$ be the cliques of $K(G_1)$, and $Y_1, Y_2, \ldots Y_m$ be the cliques of $K(G_2)$. By Lemma 2.6, there are $(n + m)$ cliques in $K(G)$. By Lemma 2.4 it follows that for every clique X_i of $K(G_1)$ there is a clique X'_i in $K(G)$, $1 \leq i \leq n$ and for every clique Y_j of $K(G_2)$ there is a clique Y'_j in $K(G)$, $1 \leq j \leq m$.

Claim 1: $X_i \cap X_j \neq \emptyset$ in $K(G_1)$ if and only if $X'_i \cap X'_j \neq \emptyset$ in $K(G)$ for $i \neq j$.

Let X_i, X_j be two cliques in $K(G_1)$ and $X_i \cap X_j \neq \emptyset$. Let v be a vertex in $X_i \cap X_j$. By Lemma 2.4 it follows that if v is a vertex in the clique X_i in $K(G_1)$, then for any vertex u in $K(G_2)$, $v + u$ is a vertex in the clique X'_i in $K(G)$ corresponding to the clique X_i in $K(G_1)$. Therefore $v + u$ is a vertex in $X'_i \cap X'_j$.

Conversely, suppose that X'_i, X'_j be two cliques in $K(G)$ and $X'_i \cap X'_j \neq \emptyset$. Let w be a vertex in $X'_i \cap X'_j$. By Lemma 2.5 it follows that $w = v + u$, where v is a vertex of $K(G_1)$ and u is a vertex of $K(G_2)$. Since $w = v + u$ is a vertex of the clique X'_i in $K(G)$, it follows that v is a vertex of the clique X_i in $K(G_1)$. Similarly v is a vertex of the clique X_j in $K(G_1)$. Therefore v is in $X_i \cap X_j$.

Similarly we can prove that, $Y_i \cap Y_j \neq \emptyset$ in $K(G_2)$ if and only if $Y'_i \cap Y'_j \neq \emptyset$ in $K(G)$ for $i \neq j$.

Claim 2: $X'_i \cap Y'_j \neq \emptyset$ in $K(G)$ for $1 \leq i \leq n$, $1 \leq j \leq m$.

Let X'_i, Y'_j be two cliques in $K(G)$, $1 \leq i \leq n$, $1 \leq j \leq m$ and X_i, Y_j are the cliques in $K(G_1), K(G_2)$ corresponding to the maximal cliques X'_i, Y'_j in $K(G)$

respectively. Let v be a vertex in X_i and u be a vertex in Y_j, then by Lemma 2.4 $v+u$ be the vertex in X_i' as well as in Y_j'. Therefore $X_i' \cap Y_j' \neq \emptyset$.

Since cliques of $K(G)$, $K(G_1)$ and $K(G_2)$ are the vertices of $K^2(G)$, $K^2(G_1)$ and $K^2(G_2)$ respectively, by claims 1 and 2 it follows that $K^2(G)$ is the same as $K^2(G_1)+K^2(G_2)$. ■

Let G_1, G_2 be two graphs, $G = G_1 + G_2$ and $K^n(G_1)$, $K^m(G_2)$ are not complete for any n, m in $\mathbb{N}$. Since $K^n(G_1)$, $K^m(G_2)$ are not complete for any n, m in $\mathbb{N}$, $K(G_1)$, $K(G_2)$ are not complete. By Theorem 2.7, $K^2(G) = K^2(G_1)+K^2(G_2)$. Since $K^n(G_1)$, $K^m(G_2)$ are not complete for any n, m in $\mathbb{N}$, $K^3(G_1) = K(K^2(G_1))$, $K^3(G_2) = K(K^2(G_2))$ are not complete. Hence by Theorem 2.7 it follows that $K^2(K^2(G)) = K^2(K^2(G_1)) + K^2(K^2(G_2))$. i.e., $K^4(G) = K^4(G_1) + K^4(G_2)$. Proceeding like this we get $K^{2n}(G) = K^{2n}(G_1) + K^{2n}(G_2)$ for any n in $\mathbb{N}$.

Theorem 2.8 *Let G_1, G_2 be two graphs and $G = G_1 + G_2$. If $K^n(G_1)$, $K^m(G_2)$ are not complete for any n, m in $\mathbb{N}$, then G is K-convergent if and only if G_1, G_2 are K-convergent.*

Proof: Let $G = G_1+G_2$ be a graph such that $K^n(G_1)$ and $K^m(G_2)$ are not complete for any n, m in $\mathbb{N}$.

Suppose G is K-convergent and G_1, G_2 are not K-convergent. By Theorem 2.7 it follows that $K^{2n}(G) = K^{2n}(G_1) + K^{2n}(G_2)$ for any n in $\mathbb{N}$. Since G_1, G_2 are not K-convergent, by definition of convergence, $K^{2n}(G_1)$ and $K^{2n}(G_2)$ are also not K-convergent for any n in $\mathbb{N}$. Therefore $K^{2n}(G)$ is not convergent for any n in $\mathbb{N}$ which is a contradiction, as if G is convergent, then $K^n(G)$ is also convergent for any n in $\mathbb{N}$.

Conversely, suppose that G_1, G_2 are K-convergent. By Theorem 2.7 it follows that $K^{2n}(G) = K^{2n}(G_1) + K^{2n}(G_2)$ for any n in $\mathbb{N}$. Since G_1, G_2 are K-convergent, by definition of convergence, the sets $\{K^n(G_1) : n \in \mathbb{N}\}$, $\{K^m(G_2) : m \in \mathbb{N}\}$ are finite, which implies that the set $\{K^{2n}(G) = K^{2n}(G_1) + K^{2n}(G_2) : n \in \mathbb{N}\}$ is also finite. i.e., there exists an n in $\mathbb{N}$ such that $K^{2n}(G) = K^{2m}(G)$ for some $m < n$, which implies that the set $\{K^n(G) : n \in \mathbb{N}\}$ is also finite. Therefore G is K-convergent. ■

Theorem 2.9 *Let G_1 , G_2 be two graphs and $G = G_1 + G_2$. If $K^n(G_1)$ is complete for some n in $\mathbb{N}$, then G is K-convergent.*

Proof: Let $G = G_1 + G_2$ be a graph. Suppose that $K^n(G_1)$ is complete for some n in $\mathbb{N}$. By Theorem 2.7 it follows that $K^{2n}(G) = K^{2n}(G_1) + K^{2n}(G_2)$ for any n in

$\mathbb{N}$. If n is even, it follows that $K^n(G) = K^n(G_1) + K^n(G_2)$. Since $K(K^n(G_1)) = K_1$ is complete, by Theorem 2.3 it follows that $K(K^n(G))$ is also complete. If n is odd, then $n+1$ is even, therefore $K^{n+1}(G) = K^{n+1}(G_1) + K^{n+1}(G_2)$. Since $K^n(G_1)$ is complete, for any $m > n$, $K^m(G_1) = K_1$ is complete. By Theorem 2.3 it follows that $K(K^{n+1}(G))$ is also complete. By the definition of clique convergence it follows that G is K-convergent. ■

Theorem 2.10 *Let G_1, G_2 be K-periodic graphs. If $G = G_1 + G_2$, then G is K-periodic.*

Proof: Let $G = G_1 + G_2$ where G_1, G_2 are K-periodic graphs of periods n, m respectively. Since G_1, G_2 are K-periodic, neither $K^i(G_1)$ nor $K^j(G_2)$ are complete for any i, j. By Theorem 2.7 it follows that

$$\begin{aligned} K^{2nm}(G) &= K^{2nm}(G_1) + K^{2nm}(G_2) \\ &= G_1 + G_2 \\ &= G \end{aligned}$$

Therefore G is K-periodic. ■

2.1 Observations

Let $G = G_1 + G_2$ be a graph and $\mathcal{K}_{G_1} = \{X_1, X_2, \ldots, X_n\}$ be the set of all cliques of G_1 and $\mathcal{K}_{G_2} = \{Y_1, Y_2, \ldots, Y_m\}$ be the set of all cliques of G_2. By Theorem 2.1, it follows that $\mathcal{K}_G = \{Q_{ij} = X_i + Y_j : 1 \leq i \leq n; 1 \leq j \leq m\}$ is the set of all cliques of G. Let v_{ij} be the vertex of $K(G)$ corresponding to the clique Q_{ij} of G. Arrange the vertices of $K(G)$ as a matrix $M = [m_{ij}]$, where $m_{ij} = v_{ij}$, i.e.,

$$\text{M=}\begin{pmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1m} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & v_{n3} & \cdots & v_{nm} \end{pmatrix}.$$

From the above matrix one can observe that the i^{th} row corresponds to the clique X_i of G_1 and j^{th} column corresponds to the clique Y_j of G_2, $1 \leq i \leq n$, $1 \leq j \leq m$.

Claim 1: Any two elements in the same row or same column in M are adjacent in $K(G)$.

Let Q_{ij}, Q_{ik} be any two elements in the i^{th} row. Since $Q_{ij} = X_i + Y_j$, $Q_{ik} = X_i + Y_k$, $Q_{ij} \cap Q_{ik} = X_i \neq \emptyset$. Therefore Q_{ij}, Q_{ik} are adjacent in $K(G)$. Similarly any two elements in the same column are adjacent.

Claim 2: If $X_i \cap X_j \neq \emptyset$, then every vertex of i^{th} row is adjacent to every vertex of j^{th} row, $1 \leq i \neq j \leq n$.

Let $X_i \cap X_j \neq \emptyset$ and v_{ik}, v_{jl} be any two elements of i^{th} and j^{th} rows respectively in M. Since $Q_{ik} = X_i + Y_k$, $Q_{jl} = X_j + Y_l$ are the cliques of G corresponding to the vertices v_{ik}, v_{jl} of $K(G)$ and $X_i \cap X_j \neq \emptyset$, we have $Q_{ik} \cap Q_{jl} \neq \emptyset$. Therefore v_{ik}, v_{jl} are adjacent in $K(G)$.

Similarly if $Y_i \cap Y_j \neq \emptyset$, then every vertex of i^{th} column is adjacent to every vertex of j^{th} column, $1 \leq i \neq j \leq m$.

One can see that the following observations will follow from Case 1 and Case 2.

1. If $G = G_1 + G_2$, then $K(G)$ is Hamiltonian.

2. If $G = G_1 + G_2$, then $K(G)$ is planar if it satisfies one of the following:

i). The number of cliques in G_1 and G_2 is less than 3.

ii). If the number of cliques in G_1 is 3, then either G_2 is a complete graph or G_2 has exactly two cliques and $K(G_1) = \overline{K_3}$, $K(G_2) = \overline{K_2}$.

iii). If the number of cliques in G_1 is 4, then G_2 is a complete graph.

3. If $G = G_1 + G_2$ and n, m are the number of cliques in G_1, G_2, then the degree of any vertex in $K(G)$ is either $(n + m - 2) + k(n - 1)$, or $(n + m - 2) + l(m - 1)$, $0 \leq k \leq m$ and $0 \leq l \leq n$.

4. Let G_1, G_2 be two graphs and $G = G_1 + G_2$,

i) If both G_1 and G_2 have odd number of cliques, then $K(G)$ is Eulerian.

ii) If both G_1 and G_2 have even number of cliques, then $K(G)$ is Eulerian if $K(G_1)$, $K(G_2)$ are Eulerian.

iii) If G_1 has even number of cliques and G_2 has odd number of cliques, then $K(G)$ is Eulerian if degree of each vertex in $K(G_1)$ is odd and $K(G_2)$ is totally disconnected.

3 Cartesian product of graphs

Theorem 3.1 *If G_1, G_2 are Clique-Helly graphs different from K_1 and $G = G_1 \square G_2$, then $K^2(G) = G$.*

Proof: Let G_1, G_2 be Clique-Helly graphs different from K_1 and $G = G_1 \Box G_2$. Let $V(G_1) = \{v_1, v_2, \ldots v_{n_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots u_{n_2}\}$, then by the definition of $G_1 \Box G_2$, it follows that $V(G) = \{V_{ij} : V_{ij} = (v_i, u_j)$ where $1 \leq i \leq n_1, 1 \leq j \leq n_2\}$, $|V(G)| = n_1 n_2$. Also, G has n_2 copies of G_1 (say, $G_1^1, G_1^2, \ldots, G_1^{n_2}$) are vertex disjoint induced subgraphs and n_1 copies of G_2 (say, $G_2^1, G_2^2, \ldots, G_2^{n_1}$) are vertex disjoint induced subgraphs. Clearly one can observe that $V(G_2^i) \cap V(G_1^j) = V_{ij}$, V_{ij} is not in $V(G_2^n)$ and $V(G_1^m)$ for $n \neq i$, $m \neq j$ for all $1 \leq i \leq n_1$, $1 \leq j \leq n_2$. As $G = G_1 \Box G_2$, we can see that every clique in G_1 and G_2 are cliques in G. Let $\mathcal{K}_{G_1} = \{Q_1, Q_2, \ldots, Q_{l_1}\}$ and $\mathcal{K}_{G_2} = \{P_1, P_2, \ldots, P_{l_2}\}$, then

$\mathcal{K}_G = \{Q_1^1, Q_2^1, \ldots, Q_{l_1}^1, Q_1^2, Q_2^2, \ldots, Q_{l_1}^2, \ldots Q_1^{n_2}, Q_2^{n_2}, \ldots, Q_{l_1}^{n_2}, P_1^1, P_2^1, \ldots, P_{l_2}^1, P_1^2, P_2^2, \ldots, P_{l_2}^2, \ldots, P_1^{n_1}, P_2^{n_1}, \ldots, P_{l_2}^{n_1}\}$.

Claim 1: For every vertex V_{ij} in G there is a clique in $K(G)$.

Let V_{ij} be a vertex in G for some i, j, $1 \leq i \leq n_1, 1 \leq j \leq n_2$. Define $A_{ij} = \{Q : V_{ij} \in Q\} \subseteq \mathcal{K}_G$. Clearly intersection of any two cliques in A_{ij} is non empty. Therefore the vertices corresponding to these cliques in $K(G)$ form a complete subgraph in $K(G)$. Suppose it is not a maximal complete subgraph in $K(G)$, then there exists a vertex V in $K(G)$ such that V is adjacent to all the vertices of A_{ij}. Let Q_V be the clique in G corresponding to the vertex V in $K(G)$. Clearly V_{ij} is not in Q_V. Since every clique in G is either a clique in G_1 or a clique in G_2, assume that Q_V is a clique in G_1^j. Let Q be a clique in G_2^i having the vertex V_{ij}, then Q is in A_{ij}. Since $V(G_2^i) \cap V(G_1^j) = V_{ij}$, Q is a clique in G_2^i and $V_{ij} \in V(Q)$ and $V(Q) \cap V(G_1^j) = V_{ij}$. Which implies that $V(Q) \cap (V(G_1^j) \setminus \{V_{ij}\}) = \emptyset$. Since V_{ij} is not in Q_V and Q_V is a clique in G_1^j, $V(Q_V) \subseteq (V(G_1^j) \setminus V_{ij})$. Therefore $V(Q) \cap V(Q_V) = \emptyset$, a contradiction to the fact that Q_V is adjacent to all the vertices of A_{ij} in $K(G)$. Hence the elements of A_{ij} form a clique in $K(G)$.

Claim 2: For any clique Q in $K(G)$, intersection of all the cliques of G corresponding to the vertices of Q is non empty and a singleton.

Let Q be a clique in $K(G)$ and $V(Q) = \{x_1, x_2, \ldots x_n\}$. Suppose all x_k's are cliques in G_1^j for some j, $1 \leq j \leq n_2$, then the intersection of all x_k's is non empty in G, where $x_k \in V(Q)$, as G_1^j satisfies clique-helly property. Let $V \in \cap_{x_k \in Q} x_k$, then V is in G_2^i for some i, $1 \leq i \leq n_1$. Let P be any clique in G_2^i having a vertex V, then P intersects with every element of $V(Q)$. Therefore $V(Q) \cup \{P\}$ forms a complete graph in $K(G)$, a contradiction to the assumption that Q is maximal complete subgraph. Thus the elements of Q are the cliques of G_1 and cliques of G_2. Since G_1^j's are vertex disjoint and G_2^i's are vertex disjoint, any element of Q is either a clique of G_1^j or a

clique of G_2^i for some fixed i, j, $1 \leq i \leq n_1$, $1 \leq j \leq n_2$. Let $x_1, x_2, \ldots, x_l$ be the cliques of G_1^j and $x_{l+1}, x_{l+2}, \ldots, x_n$ be the cliques of G_2^i. Since $V(G_1^j) \cap V(G_2^i) = V_{ij}$, x_{l_1} is a clique of G_1^j, x_{l_2} is a clique of G_2^i and $V(x_{l_1}) \cap V(x_{l_2}) \neq \emptyset$, $1 \leq l_1 \leq l$, $l+1 \leq l_2 \leq n$, $V(x_{l_1}) \cap V(x_{l_2}) = V_{ij}$. Which implies that V_{ij} belongs to every x_k in Q. Therefore $\cap_{x_k \in Q} x_k = V_{ij}$.

As the cliques of $K(G)$ are the vertices of $K^2(G)$, by Claims 1 and 2 one can see that there is a one to one correspondence between the vertices of G and $K^2(G)$.

Claim 3: Let U, V be any two adjacent vertices in G. Then the intersection of the cliques in $K(G)$ corresponding to these vertices is non empty.

Let U, V be any two adjacent vertices in G and Q_U, Q_V be the cliques in $K(G)$ corresponding to the vertices U, V in G respectively. Since there is an edge between U, V in G, there exists a clique Q in G such that the vertices U, V are in Q. By Claims 1 and 2 it follows that the vertices of Q_U in $K(G)$ are the cliques of G having the vertex U in G is in common. Therefore Q is in $V(Q_U)$. Similarly Q is in $V(Q_V)$. Which implies that $Q_U \cap Q_V \neq \emptyset$. Since cliques of $K(G)$ are the vertices of $K^2(G)$, the vertices corresponding to the cliques Q_U and Q_V of $K(G)$ are adjacent in $K^2(G)$.

Claim 4: Let P, Q be any two cliques in $K(G)$. If the intersection of P and Q is non empty, then the vertices in G corresponding to these two cliques are adjacent.

Let P, Q be any two cliques in $K(G)$, $P \cap Q \neq \emptyset$ and U, V be the vertices in G corresponding to the cliques P, Q of $K(G)$ respectively. Since $P \cap Q \neq \emptyset$, there exists a vertex Q_1 belonging to $V(P) \cap V(Q)$. By Claims 1 and 2, one can observe that Q_1 is a clique in G and $\cap_{P_i \in V(P)} P_i = U$, $\cap_{Q_i \in V(Q)} Q_i = V$. Thus U, V belongs to $V(Q_1)$ in G. Therefore U, V are adjacent in G.

By Claims 3 and 4 it follows that, two vertices are adjacent in G if and only if the corresponding vertices are adjacent $K^2(G)$.

Therefore $K^2(G)$ is the same as G, if $G = G_1 \Box G_2$ and G_1, G_2 are Clique-Helly graphs such that G_1, G_2 are different from K_1. ■

Corollary 3.2 *Let G_1, G_2 be two graphs and $G = G_1 \Box G_2$. If G_1, G_2 are Clique-Helly graphs different from K_1, then*

i *G is a Clique-Helly graph.*

ii *G is K-periodic.*

iii *G is K-convergent.*

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